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Simultaneous control of linear systems by Genetic Algorithms in state and output feedback

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Abstract

In this paper, Genetic Algorithms (GAs) are employed to control simultaneous linear systems in both state and output feedback. First, the similarity transformation is applied to obtain parameterized controllers. This requires solution of a system of equations and also some non-linear inequalities. GAs are used to solve these equations and inequalities. Therefore, the paper presents an analytical method for finding parameterized controllers and employs a numerical method to enhance the solution. Three numerical examples are presented to illustrate the effectiveness of the method and to compare the results with previous results.

Keywords*:* Linear Systems; Genetic Algorithms; simultaneous control; optimization; eigenvalue assignment

1. Introduction

The problem of simultaneous stabilization of time invariant linear systems

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t)\ ,\ k = 1,2,\ldots,p$$
$$y_k(t) = C_k x_k(t) \tag{1}$$

is to find a state feedback controller matrix F (in case of $C_k = I$) with the feedback law $u_k = Fx_k(t)$ or to find a output feedback controller matrix K with the feedback law $u_k = Ky_k(t)$, such that the eigenvalues of all closed loop systems $A_{kc} = A_k + B_k F$ or $A_{kc} = A_k + B_k K C_k$ for all $k = 1, 2, \ldots, p$ lie in the left hand side of the complex plane in the prescribed bounded region.

Investigation into this problem was first introduced by Saeks and Murray [1], based on the work of Birdwell et al [2]. In the case of two plants, the simultaneous stabilization problem reduces to a well-known problem [3] and a proper stable controller is found to stabilize both plants. However, simultaneous stabilization of more than two plants, in general, is difficult [4]. An analytical solution to the simultaneous stabilization problem is NP-hard to find [5] and so no analytical algorithm can be devised to lead to a simple or rapid solution. Furthermore, the numerical methods are replaced instead of the analytical methods. Such approaches had focus in two main areas. The first deals with the solution to the simultaneous stabilization problem itself, in which the solution to the problem has been tackled from a number of different directions, including polynomials [6], minimum phase [7], system inversion [8], optimization-based [9], similarity transformation [10] and decomposition-based [11]. The second deals with performance improvement for the simultaneous stabilization, or simultaneous optimal control, in which nonlinear optimization algorithms have been proposed [12].

Simultaneous stabilization problem of a finite collection of distinct systems under a single feedback controller in many engineering problems is important, such as control of aircraft [13], particularly when different conditions are produced by dynamical models. A single stabilization control provides system simplicity and reliability. For example, it could be used as a backup reconfigurable control under actuation system damage or failure [11].

In [13], a nonlinear state feedback controller which simultaneously stabilizes a collection of single input systems is presented. In [14], necessary and sufficient conditions embedded in the solvability of a constrained optimization problem for the existence of controllers to simultaneously stabilize a collection of single input–multi output systems are obtained. In [15], the method of [14] was modified for both output and state feedback. In [16] the problem was considered only for single

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input systems. In [17] the optimal simultaneous state feedback controller by numerical solution of a minimizing problem is obtained. In [18] an auxiliary minimization problem for computing an approximate solution instead of original problem is solved.

In these papers, an enormous amount of computation is needed to find a simultaneous controller, usually resulting in a big norm when intelligent methods are not implemented. One of the intelligent methods used is GAs. This method was first applied for assigning eigenvalues in state feedback control [19]. Artificial intelligence, like GAs is beneficial when analytical methods fail. The GAs consider all the constraints on the controller matrix simultaneously and obtain local optimal points [19]. GAs have the method of global search in natural selection of genes. They act on existing individual population that has been selected randomly in the beginning of the search in order to improve the solution.

In this paper, a new method for computing simultaneous state feedback and output feedback for eigenvalue assignment of a collection of linear systems in a region is presented. A set of equations and inequalities are obtained and then are transformed to an optimization problem. Enormous computational power of GAs is implemented for solving this kind of optimization problem. Here, first by using GAs and an objective function depending on eigenvalues, an optimal problem is solved for finding a controller matrix. A fitness function satisfying the optimal value of the set of equations and the set of inequalities is introduced. Then, by adjusting other parameters of GAs the solution is obtained. Comparing our work with previous methods, results in simpler computations using an intelligent method of GAs. Indeed, using Riccati equations [18], cost functions and decomposition-based methods [11] requires very complicated computations, whereas using the algorithm presented here is much more straight forward.

The structure of the paper is as follows. In the next section, formulation of the problem is presented. Also, similarity transformations for finding state and output feedback matrices are introduced. In the third section, a method based on optimization for solving simultaneous system of linear equations and non-linear inequalities is introduced. Also, the resulting method is summarized in an algorithm. Finally, some examples are illustrated in order to show the effectiveness of the presented method and a comparison is made with the previous results in [18] and [11].

2. Problem formulation

Consider a set of p time invariant systems of (1), where $x_k \in R^n$ is the state vector, $u_k \in R^m$ is input vector and $y_k \in R^r$ is the output vector of k^{th} system. A_k, B_k, C_k are constant matrices of dimensions $n \times n, n \times m$ and $r \times n$ respectively with the following assumptions:

1. (A_k, B_k) are controllable and (A_k, C_k) are observable.
2. B_k and C_k have full row ranks.

Now consider $A_{kc} = A_k + B_k F$, closed-loop systems with the state feedback control laws $u_k = F x_k$ or $A_{kc} = A_k + B_k K C_k$ closed-loop systems with the output feedback control laws $u_k = K y_k$. The objective is to find a state or output feedback matrix for all the p systems which satisfies the above assumptions, such that all the roots of the characteristic equations of each closed-loop system lie in a prescribed region. Here, it is assumed that the roots lie inside a rectangular region defined as:

$$\Omega = \{ s \in C \mid \alpha \le real(s) \le \beta \ , -\gamma \le imag(s) \le \gamma \} \quad (2)$$

where $\alpha \in R, \beta \in R$ and $\gamma \in R$. This region is considered symmetric with respect to the real axis in order to obtain a real state feedback matrix F and a real state feedback matrix K [18]. A brief review of the paper [20] is recalled for the computation of the state and output feedback matrix.

2.1. Similarity Transformation Method

An existing and analytical method of finding a state and an output feedback matrix by similarity transformations is given in [21]. For computing state feedback matrix F_k and output feedback matrix K_k for all the p systems, first the augmented matrix $[B_k, A_k, I_n]$ is transformed to vector companion form $[\tilde{B}_k, \tilde{A}_k, T_k^{-1}]$ by elementary similarity operations [20]. Then state feedback and output feedback matrix can be found from:

$$F_k = B_{k0}^{-1}(-G_{k0} + G_{k\lambda})T_k^{-1} \quad (3)$$

$$K_k C_k = B_{k0}^{-1}(-G_{k0} + G_{k\lambda})T_k^{-1} \quad (4)$$

Where, B_{k0}, G_{k0} and T_k^{-1} are block matrices with appropriate dimensions and are selected from vector companion form $[\tilde{B}_k, \tilde{A}_k, T_k^{-1}]$ as [20]:

$$\tilde{B}_k = \begin{bmatrix} B_{k0} \\ 0_{n-m,m} \end{bmatrix}$$
$$\tilde{A}_k = \begin{bmatrix} G_{k0} \\ I_{n-m}, 0_{n-m,m} \end{bmatrix} \tag{5}$$

Let the parametric closed-loop matrix of each system with the desired eigenvalues be $\tilde{\Gamma}_{k\lambda}$ in the following form:

$$\tilde{\Gamma}_{k\lambda} = \begin{bmatrix} G_{k\lambda} \\ I_{n-m}, 0_{n-m,m} \end{bmatrix}, \ where \ \ G_{k\lambda} = \begin{bmatrix} g_{k11} \ g_{k12} \cdots g_{k1n} \\ g_{k21} \ g_{k22} \cdots g_{k2n} \\ \vdots \\ g_{km1} \ g_{km2} \cdots g_{kmn} \end{bmatrix} \tag{6}$$

In which $G_{k\lambda}$ is a parametric matrix of dimension $m \times n$ obtained from first m rows of $\tilde{\Gamma}_{k\lambda}$.

The closed-loop system eigenvalues of A_{kc}, can be located in a prescribed spectrum by $\tilde{\Gamma}_{k\lambda}$. For this reason, it is sufficient to have $\det(\tilde{\Gamma}_{k\lambda} - \lambda_k I) = 0$, which leads to the characteristic polynomial of $\tilde{\Gamma}_{k\lambda}$ as:

$$\det(\tilde{\Gamma}_{k\lambda} - \lambda_k I) = P_{kn}(\lambda_k) \tag{7}$$

where

$$P_{kn}(\lambda_k) = (-1)^n \left(\lambda_k^n + c_{k1}\lambda_k^{n-1} + \ldots + c_{k(n-1)}\lambda_k + c_{kn} \right) \tag{8}$$

is the characteristic polynomial of the closed-loop system.

Since it is necessary that all the roots of the characteristic polynomial lie in the spectrum $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn}\}$, it is clear that:

$$P_{kn}(\lambda_k) = (-1)^n (\lambda_k - \lambda_{k1})(\lambda_k - \lambda_{k2})\ldots(\lambda_k - \lambda_{kn}) \tag{9}$$

By equating the above equalities, $c_{ki}, (i = 1, 2, \ldots, n)$ can be computed [20] as:

$$\begin{aligned} c_{k1} &= -\sum_{i=1}^{n} (\lambda_{ki}) \\ c_{k2} &= \sum_{i,j=1, i\neq j}^{n} (\lambda_{ki}\lambda_{kj}) \\ &\vdots \\ c_{kn} &= (-1)^n \prod_{i=1}^{n} (\lambda_{ki}) \end{aligned} \tag{10}$$

If $\lambda_{ki}, (i = 1, 2, \ldots, n)$ are known, then $c_{k1}, c_{k2}, \ldots, c_{kn}$ can be found. Now, with direct computation of $\det(\tilde{\Gamma}_{k\lambda} - \lambda_k I) = P_{kn}(\lambda_k)$ parametrically and with having coefficients of characteristic polynomial in equation (10), a set of system of non-linear equations results, as follows:

$$\begin{aligned} &f_{k1}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) = c_{k1} \\ &f_{k2}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) = c_{k2} \\ &\vdots \\ &f_{kn}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) = c_{kn} \end{aligned} \tag{11}$$

where, $g_{kij}, (i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n)$ are the elements of $G_{k\lambda}$ and $f_{ki}, (i = 1, 2, \ldots n)$ are parametric non-linear polynomials that are obtained by computing $\det(\tilde{\Gamma}_{k\lambda} - \lambda_k I)$. The set of equations (11) is a set of non-linear system with n equations and nm unknowns. By arbitrary selection of $N = n(m-1)$ unknowns this system can be solved.

2.2. Finding a simultaneous state feedback matrix

In this section, we introduce a new method for computing a simultaneous state feedback by similarity transformations for a collection of controllable systems. Consider the given systems (1). For the systems to have a simultaneous state feedback matrix, the equation (3) must be changed to:

$$F = B_{k0}^{-1}(-G_{k0} + G_{k\lambda})T_k^{-1}, \ for\ each \ \ k = 1, 2, \ldots, p \tag{12}$$

Here, p equations are obtained and by equating them together other equations are derived. For example, if $k = i$ and then $k = j$ is considered, it follows that:

$$F = B_{j0}^{-1}(-G_{j0} + G_{j\lambda})T_j^{-1} = B_{i0}^{-1}(-G_{i0} + G_{i\lambda})T_i^{-1} \tag{13}$$

From which

$$G_{i\lambda} - B_{i0}B_{j0}^{-1}G_{j\lambda}T_j^{-1}T_i = G_{i0} - B_{i0}B_{j0}^{-1}G_{j0}T_j^{-1}T_i \tag{14}$$

Here, unknowns of the equations are $G_{i\lambda}, G_{j\lambda}$. The remaining matrices are computed from transformation of pairs (A_i, B_i) and (A_j, B_j) into vector companion form as mentioned before. Hence, the left hand side of (14) is an unknown matrix of dimensions $m \times n$ and the right hand side is a known matrix of dimensions $m \times n$. Now, by equating the corresponding elements of these matrices, mn equations with $2mn$ unknowns are obtained. Finally, by equating the right hand side of (12) term by term for each $i = 1$, say, and $j = 2,..., p$, $(p-1)mn$ equations in the form of (14), with mnp unknowns are derived.

Solution of these equations results in a state feedback matrix, but does not guarantee the stability of the systems under consideration. For this reason, other constraints must be considered such that the systems are stabilized. In order to stabilize the systems, the defined region in (2) must lie in the left hand side of the complex plane, so that the eigenvalues of systems (1) lie inside this region. The equations of (11) guarantee eigenvalue assignment of systems (1) in a prescribed spectrum. Although the new method does not allocate the eigenvalues exactly, it can assign the eigenvalues in a prescribed rectangular symmetric bounded region with respect to the real axis. Hence, upper bounds and lower bounds for the left hand side of (11) are considered where: $c_{i\max}$ are the upper bounds and $c_{i\min}, (i = 1, 2, \ldots, n)$ are the lower bounds as given below:

$$
\begin{aligned}
&c_{1\max} = -n(\lambda_{\max}) \\
&c_{2\max} = (-1)^2 n(\lambda_{\max}^2) \qquad \textit{and} \\
&\vdots \\
&c_{n\max} = (-1)^n (\lambda_{\max}^n) \\
&c_{1\min} = -n(\lambda_{\min}) \\
&c_{2\min} = (-1)^2 n(\lambda_{\min}^2) \\
&\vdots \\
&c_{n\min} = (-1)^n (\lambda_{\min}^n)
\end{aligned}
\tag{15}
$$

where $\lambda_{\max}$ is set to α and $\lambda_{\min}$ is set to β as introduced in (2). In this case, equations (10) are transformed to the following inequalities:

$$
\begin{aligned}
&c_{1\min} \le f_{k1}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) \le c_{1\max} \\
&c_{2\min} \le f_{k2}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) \le c_{2\max} \\
&\vdots \\
&c_{n\min} \le f_{kn}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) \le c_{n\max n}
\end{aligned}
\tag{16}
$$

These inequalities can be rewritten in the form:

$$
\begin{aligned}
&f_{k1}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) - c_{1\max} \le 0 \\
&f_{k2}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) - c_{2\max} \le 0 \\
&\vdots \\
&f_{kn}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) - c_{n\max} \le 0 \\
&-f_{k1}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) + c_{1\min} \le 0 \\
&-f_{k2}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) + c_{2\min} \le 0 \\
&\vdots \\
&-f_{kn}(g_{k11}, g_{k12}, \ldots, g_{k1n}, g_{k21}, g_{k22}, \ldots, g_{k2n}, \ldots, g_{km1}, g_{km2}, \ldots, g_{kmn}) + c_{n\min} \le 0
\end{aligned}
\tag{17}
$$

where now, $2np$ inequalities with nmp unknowns are obtained. By using the set of equalities (14) and the set of inequalities (17) and by solving them simultaneously, a vector $g \in R^{nmp}$ whose elements are defined in (17) can be found such that a simultaneous state feedback matrix which stabilizes systems (1) is obtained.

2.3. Finding a simultaneous output feedback matrix

In this section, a new method for computing a simultaneous output feedback by similarity transformations for a collection of controllable systems is introduced. Consider the given systems (1). For the systems to have a simultaneous output feedback, equation (4) should be changed to:

$$
KC_k = B_{k0}^{-1}(-G_{k0} + G_{k\lambda})T_k^{-1}, \textit{ for each } k = 1, 2, \ldots, p \tag{18}
$$

The left hand sides of (18) are not equal for each of the p equations. Therefore, the right hand sides of (18) for each of the p equations cannot be equated as it was done in the previous section. To overcome this problem, elementary similarity operations on the pairs (C_k, I_n) are performed in order to obtain $(\tilde{C}_k, E_k)$ such that [21]:

$$
\tilde{C}_k = C_k E_k = \begin{bmatrix} I_r & 0_{r,n-r} \end{bmatrix}, \; k = 1, 2, \ldots, p \tag{19}
$$

Multiplying (18) by E_k on the right hand side yields:

$$
\begin{aligned}
KC_k E_k &= B_{k0}^{-1}(-G_{k0} + G_{k\lambda})T_k^{-1} E_k \\
&= \begin{bmatrix} K & 0_{r,n-r} \end{bmatrix}, \; k = 1, 2, \ldots, p
\end{aligned}
\tag{20}
$$

As a result, now the p middle relations of (20) can be equated to each other. For example, if $k = i$ and then $k = j$ is considered, the following equations will be derived:

$$
B_{j0}^{-1}(-G_{j0} + G_{j\lambda})T_j^{-1}E_j = B_{i0}^{-1}(-G_{i0} + G_{i\lambda})T_i^{-1}E_i = \begin{bmatrix} K & 0_{r,n-r} \end{bmatrix} \tag{21}
$$

From which:

$$G_{i\lambda} - B_{i0}B_{j0}^{-1}G_{j\lambda}T_j^{-1}E_jE_i^{-1}T_i = G_{i0} - B_{i0}B_{j0}^{-1}G_{j0}T_j^{-1}E_jE_i^{-1}T_i \quad (22)$$

Now, the left hand side is an unknown matrix of dimensions $m \times n$ and the right hand side is a known matrix of dimensions $m \times n$. Hence, by equating the corresponding elements, mn equations with $2mn$ unknowns are obtained. Finally, taking $k = 1,2,\ldots,p$ results in $(p-1)mn$ equations and as in (22) with mnp unknowns. Solution of these equations results in an output feedback matrix, but does not guarantee the stability of controlled systems as before. However, for stabilizing the system the following approach is devised.

Let all the n eigenvalues of p systems in (1) be located in region (2) in the form:

$$\begin{aligned} &\Omega_k = \{\alpha_{k1} + i\,\beta_{k1}, \alpha_{k2} + i\,\beta_{k2}, \ldots, \alpha_{kn} + i\,\beta_{kn}\} \\ &k = 1,2,\ldots,p \end{aligned} \quad (23)$$

Here, α_{kj}, β_{kj}, $j = 1,2,\ldots,n$ are respectively real parts and imaginary parts of corresponding eigenvalues of the closed-loop systems of (1). For eigenvalues in (23) to lay in the region (2) the following inequalities for $k = 1,2,\ldots,p$ must be satisfied:

$$\begin{aligned} h_{ki} &= \alpha_{ki} - \eta \le 0 \quad i = 1,2,\ldots,n \\ h_{kj} &= \delta - \alpha_{kj} \le 0 \quad j = n+1, n+2,\ldots,2n \\ h_{kr} &= \beta_{kr} - \gamma \le 0 \quad r = 2n+1, 2n+2,\ldots,3n \\ h_{ks} &= -\gamma - \beta_{ks} \le 0 \quad s = 3n+1, 3n+2,\ldots,4n \end{aligned} \quad (24)$$

In what follows a method for solving simultaneously the set of equations (14) and inequalities (17) for finding a state feedback matrix or for solving simultaneously the set of equations (22) and inequalities (24) for finding an output feedback matrix is presented.

3. A method of solution

For solving systems of equations (14) and the systems of inequalities (17) or systems of equations (22) and the systems of inequalities (24) simultaneously, the following lemma facilitates the solution.

Lemma 3.1. The two systems I and II introduced below are equivalent:

$$I \quad \begin{cases} j_1(x) = 0 \\ j_2(x) = 0 \\ \vdots \\ j_k(x) = 0 \\ h_1(x) \le 0 \\ h_2(x) \le 0 \\ \vdots \\ h_l(x) \le 0 \end{cases} \qquad \textit{and}$$

$$I \quad \begin{cases} \min \quad j_1^2(x) + j_2^2(x) + \ldots + j_k^2(x) \\ \quad st \\ \quad\quad h_1(x) \le 0 \\ \quad\quad h_2(x) \le 0 \\ \quad\quad \vdots \\ \quad\quad h_l(x) \le 0 \end{cases} \quad (25)$$

if and only if the object function in system II is zero for the optimal point.

Proof: Let

$$X = \left\{ x \in R^n \,\middle|\, h_1(x) \le 0, h_2(x) \le 0, \ldots, h_l(x) \le 0 \right\} \quad (26)$$

If the system I has a feasible solution x_0, then

$$\exists x_0 \in R^n \ni j_1(x_0) = j_2(x_0) = \ldots = j_k(x_0) = 0 \quad (27)$$

$$h_1(x_0) \le 0, h_2(x_0) \le 0, \ldots, h_l(x_0) \le 0 \quad (28)$$

From (28) it results that $x_0 \in X$ and from (27)

$$j_1^2(x_0) + j_2^2(x_0) + \ldots + j_k^2(x_0) = 0 \quad (29)$$

But since the object function of system II is non-negative, it takes its minimal value at zero, hence, x_0 is a feasible solution for system II.

Conversely, if the optimized solution of system II is zero, then the system I also has a solution and this completes the proof.

For solving simultaneously the system of equations (14) and the inequalities (17) or the system of equations (22) and the inequalities (24), the lemma 3.1 may be used. For this purpose, $J_i(x)$ in system II must be replaced by the equations introduced in (14) or in (22), and also the constraints of system II must be replaced by the inequalities (17) or (24). Then the fitness function is obtained by linear combination of object function and constraints of system II which can be solved by GAs. Here the number of equations and inequalities are $k=(p-1)mn$ and $l=4np$, respectively.

3.1. Selecting proper fitness function

For each p system defined in (1), the eigenvalues can be assigned in a bounded region or in a prescribed spectrum [21]. For assigning the eigenvalues of a collection of controllable systems in a prescribed region for simultaneous stabilization, the systems of inequalities (17) or (24) must be satisfied according to state or output feedback control.

In using GAs for solving the set of equations (14) and inequalities (17) or the set of equations (22) and inequalities (24), it is necessary to apply a proper fitness function. To simplify the fitness function, a vector H is defined in the form:

$$H=(h_{11},h_{12},\ldots,h_{1(4n)},h_{21},h_{22},\ldots,h_{2(4n)},\ldots,h_{p1},h_{p2},\ldots,h_{p(4n)})$$
$$X=\left\{x\in R^n \,\middle|\, h_1(x)\le 0,h_2(x)\le 0,\ldots,h_l(x)\le 0\right\} \quad (30)$$

where $h_{ki}, k=1,2,\ldots,p, i=1,2,\ldots,4n$ are the constraints obtained from system II, and

$$S=\sum_{k=1}^{p}\sum_{i=1}^{4n}(sign(h_{ki})+1) \quad (31)$$

is defined in order to transform the inequalities of system II into a non-negative number. In order to satisfy equation (31) considering condition (2) it is necessary that S be zero. Fitness function is obtained from nonlinear combination of (14) or (22) with (31) in the form:

$$Y=j_1^2(g)+j_2^2(g)+\ldots+j_{(p-1)mn}^2(g)+S \quad (32)$$

where j_s, $s=1,2,\ldots,(p-1)mn$ are equations obtained from (17) or (24). This fitness function is implemented in GAs to solve the optimization problem in mind. The feasible solution corresponding to fitness function is a vector $g\in R^{nmp}$ whose elements are the same as the elements of $G_{k\lambda}$ matrix. By substituting elements of $G_{k\lambda}$ matrix in (12) a state feedback matrix or in (18) an output feedback matrix is obtained for simultaneous control. The above results can be summarized in the following algorithm.

3.2. The Algorithm

Object: To obtain simultaneous output feedback matrix K, for which the eigenvalues of the closed-loop systems G's are located in a prescribed spectrum.

Input: The controllable and observable system matrices (A_k,B_k,C_k) and the eigenvalue spectrums $\Omega_k=\{\alpha_{k1}+i\beta_{k1},\alpha_{k2}+i\beta_{k2},\ldots,\alpha_{kn}+i\beta_{kn}\}\quad k=1,2,\ldots,p$ where complex eigenvalues are in complex conjugate pairs.

Output: The output feedback matrix K, such that the eigenvalues of each closed-loop system fall into the prescribed spectrum.

Step 1: Employ the algorithm given by [20] to obtain B_{k0}^{-1},G_{k0} and T_k^{-1} for $k=1,2,...,p$.

Step 2: Obtain the coefficients of the characteristic polynomials whose roots are the same as the desired eigenvalue spectrums $\Omega_k=\{\alpha_{k1}+i\beta_{k1},\alpha_{k2}+i\beta_{k2},\ldots,\alpha_{kn}+i\beta_{kn}\}\quad k=1,2,\ldots,p$.

Step 3: Obtain the characteristic polynomials of $\tilde{A}_k$'s as defined in [20].

Step 4: Obtain the non-linear systems of equations relating parameters g_{kij}, by equating the coefficients of the characteristic polynomials obtained in Steps 2 and 3.

Step 5: Obtain inequalities (17).

Step 6: Use the results obtained in Steps 4 and 5 to implement the Lemma 3.1in order to find the fitness function.

Step 7: Employ the Genetic Algorithm with the fitness function obtained in Step 6 for finding $G_{k\lambda}$.

Step 8: Obtain $(\tilde{C}_k,E_k)$. The matrices E_k are obtained by elementary column operations on C_k such that $\tilde{C}_k=C_kE_k=\begin{bmatrix}I_r & 0_{r,n-r}\end{bmatrix}, k=1,2,\ldots,p$ (as in [21]).

Step 9: Substitute the results obtained in

$$\begin{aligned}KC_kE_k&=B_{k0}^{-1}(-G_{k0}+G_{k\lambda})T_k^{-1}E_k\\&=\begin{bmatrix}K & 0_{r,n-r}\end{bmatrix}, k=1,2,\ldots,p\end{aligned}$$

Step 10: Store the first r columns of the matrix obtained in step 9 to obtain K.

Using this algorithm avoids direct solution of systems of nonlinear equations which is numerically complicated; instead the output feedback matrix is obtained readily using the fitness function defined by a search method. The following examples use the above algorithm to obtain simultaneous state or output feedback matrix and the results are compared with those obtained in [18] and [11].

4. Illustrative examples

The following three examples are given to illustrate the effectiveness of the presented method. The results are used to compare the presented method with the existing methods.

Example 1. Consider the linear controllable systems given in [18]:

$$A_1=\begin{bmatrix}-0.98960 & 17.4100 & 96.15\\ 0.26480 & -0.8512 & -11.89\\ 0 & 0 & -30\end{bmatrix},\quad B_1=\begin{bmatrix}-97.78\\ 0\\ 30\end{bmatrix} \tag{33}$$

$$A_2=\begin{bmatrix}-0.66070 & 18.1100 & 84.34\\ 0.08201 & -0.6587 & -10.81\\ 0 & 0 & -30\end{bmatrix},\quad B_2=\begin{bmatrix}-272.2\\ 0\\ 30\end{bmatrix} \tag{34}$$

$$A_3=\begin{bmatrix}-1.70200 & 50.7200 & 263.50\\ 0.22010 & -1.4180 & -31.99\\ 0 & 0 & -30\end{bmatrix},\quad B_3=\begin{bmatrix}-85.09\\ 0\\ 30\end{bmatrix} \tag{35}$$

$$A_4=\begin{bmatrix}-0.51620 & 26.9600 & 178.90\\ -0.68960 & -1.2250 & -30.38\\ 0 & 0 & -30\end{bmatrix},\quad B_4=\begin{bmatrix}175.6\\ 0\\ 30\end{bmatrix} \tag{36}$$

The aim is to obtain a state feedback controller which assigns the eigenvalues inside the region:

$$\Omega=\{s\in C \mid -10\le real(s)\le -0.3\,,\ -40\le imag(s)\le 40\} \tag{37}$$

Similarity transformations are performed to obtain necessary matrices. By solving the set of equations (14) and inequalities (17) by using GAs the following state feedback matrix is obtained:

$$F=\begin{bmatrix}-0.0010 & 1.2875 & 0.7541\end{bmatrix} \tag{38}$$

The resulting eigenvalues of the corresponding closed-loop systems are found to be:

$$\begin{aligned}v_1&=\{-0.8830,-4.1185+21.8639i,-4.1185-21.8639i\}\\ v_2&=\{-1.9556,-3.2343+20.6406i,-3.2343-20.6406i\}\\ v_3&=\{-0.3946,-5.0177+35.3588i,-5.0177-35.3588i\}\\ v_4&=\{-7.4910,-0.9014+36.8502i,-0.9014-36.8502i\}\end{aligned} \tag{39}$$

As it can be verified, they are all inside the given region. It should be noted that the norm of the state feedback matrix here is 1.4921. The feedback matrix obtained in [18] is $F_{Wu}=\begin{bmatrix}0.50263 & 4.29837 & -0.40365\end{bmatrix}$, with the norm 4.3464; clearly, the norm obtained by our method is much reduced.

Example 2. Let us now consider the linear controllable and observable systems given in [18] for output feedback control:

$$A_1=\begin{bmatrix}-4 & 5 & -4\\ 4 & -21 & -18\\ -32 & -4 & 34.5\end{bmatrix},\quad B_1=\begin{bmatrix}1 & 1\\ -5 & 3\\ 0 & -4\end{bmatrix},\quad C_1=\begin{bmatrix}0 & 1 & 1\\ -2 & 0 & 3\end{bmatrix} \tag{40}$$

$$A_2=\begin{bmatrix}-43 & 24 & -4\\ -98 & 33 & 20\\ 49 & 49 & -94\end{bmatrix},\quad B_2=\begin{bmatrix}0 & 1\\ 1 & 2\\ -3 & 1\end{bmatrix},\quad C2=\begin{bmatrix}-4 & -8 & 12\\ 8 & 8 & 4\end{bmatrix} \tag{41}$$

Here the aim is to obtain an output feedback controller which assigns the eigenvalues inside the region:

$$\Omega=\{s\in C \mid -12\le real(s)\le -1\,,\ -35\le imag(s)\le 35\} \tag{42}$$

By using our method incorporating GAs, the following output feedback matrix is obtained:

$$K = \begin{bmatrix} -1.4559 & 0.6863 \\ 3.7403 & 1.9212 \end{bmatrix} \quad (43)$$

The resulting eigenvalues of the corresponding closed-loop systems are found to be:

$$\begin{aligned} v_1 &= \{-9.6314, -2.7994+15.8430i, -2.7994-15.8430i\} \\ v_2 &= \{-10.0264, -4.3940+29.5101i, -4.3940-29.5101i\} \end{aligned} \quad (44)$$

which again lie inside the prescribed region. It should be noted that the norm of the output feedback matrix here is 4.3285. However, the feedback matrix obtained in [18] is $K_{Wu} = \begin{bmatrix} -0.7183 & 1.4953 \\ 4.8755 & 2.4765 \end{bmatrix}$ with norm 5.4686.

Example 3. In the last comparison, consider the linear controllable systems given in [11]:

$$A_1 = \begin{bmatrix} -0.98960 & 17.4100 & 96.15 \\ 0.26480 & -0.8512 & -11.89 \\ 0 & 0 & -30 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix} \quad (45)$$

$$A_2 = \begin{bmatrix} -0.66070 & 18.1100 & 84.34 \\ 0.08201 & -0.6587 & -10.81 \\ 0 & 0 & -30 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -272.2 \\ 0 \\ 30 \end{bmatrix} \quad (46)$$

$$A_3 = \begin{bmatrix} -1.7202 & 50.7200 & 263.50 \\ 0.22010 & -1.4180 & -31.99 \\ 0 & 0 & -30 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix} \quad (47)$$

$$A_4 = \begin{bmatrix} -0.51620 & 26.9600 & 178.90 \\ -0.68960 & -1.2250 & -30.38 \\ 0 & 0 & -30 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -175.6 \\ 0 \\ 30 \end{bmatrix} \quad (48)$$

The aim is to obtain a state feedback controller which assigns the eigenvalues inside the region:

$$\Omega = \{ s \in C \mid -4 \le real(s) \le -1, -20 \le imag(s) \le 20 \} \quad (49)$$

By using GAs the following state feedback matrix is obtained:

$$F = \begin{bmatrix} 0.0088 & 0.3788 & 0.8861 \end{bmatrix} \quad (50)$$

The resulting eigenvalues of the closed-loop systems are thus:

$$\begin{aligned} v_1 &= \{-1.2689, -2.4247-11.6867i, -2.4247+11.6867i\} \\ v_2 &= \{-1.9155, -2.6082-13.0053i, -2.6082+13.0053i\} \\ v_3 &= \{-1.6094, -2.8473-17.5874i, -2.8473+17.5874i\} \\ v_4 &= \{-1.5265, -2.5885-17.6341i, -2.5885+17.6341i\} \end{aligned} \quad (51)$$

Once again these are inside the given region. It should be noted that the norm of the state feedback matrix here is 0.9637, while the feedback matrix obtained in [11] is $F_{Ruben} = [-0.209582 \; -1.373644 \; 1.584912]$ with the norm 2.1078.

5. Conclusion

In this paper, by using similarity transformations a suitable fitness function is obtained which is then implemented in GAs in order to find state/output feedback matrices for the simultaneous control of a collection of linear systems. This method is much simpler than the methods which employ Riccati equations [18] and cost functions and decomposition-based methods [11]. As the illustrative examples show, the results obtained have a lesser norm. The ability of GAs to find the global minimum results in better solutions for simultaneous control [22]. In general, the merit of the presented method is the simplicity of the algorithm, less amount of computational effort, the insensibility in varying the prescribed region and the reduction in the norm of feedback matrices relative to the existing methods. The GAs always leads to a feasible solution if the systems under consideration are all controllable and observable.

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